CHAMPS SCALAIRES ET CHAMPS DE VECTEURS

Les formules encadrées sont à savoir par cœur.

1 – Opérateur « nabla »

Opérateur différentiel « nabla » : $\vec{\nabla} = \left(\dfrac{\partial}{\partial x}\right).\overrightarrow{u_x} + \left(\dfrac{\partial}{\partial y}\right).\overrightarrow{u_y} + \left(\dfrac{\partial}{\partial z}\right).\overrightarrow{u_z}$ en coordonnées cartésiennes

Gradient du champ scalaire f(x,y,z) : $\overrightarrow{grad}\,f$ ou $\vec{\nabla} f$

Divergence du champ de vecteurs $\vec{A}$: $div\vec{A}$ ou $\vec{\nabla}.\vec{A}$

Rotationnel du champ de vecteurs $\vec{A}$: $\overrightarrow{rot}.\vec{A}$ ou $\vec{\nabla} \wedge \vec{A}$

Laplacien du champ scalaire f : $\Delta f = \mathrm{div}.\left(\overrightarrow{\mathrm{grad}}.f\right)$

du champ de vecteurs $\vec{A}$: $\vec{\Delta}.\vec{A} = (\Delta A_x).\overrightarrow{u_x} + (\Delta A_y).\overrightarrow{u_y} + (\Delta A_z).\overrightarrow{u_z}$

2 – Formulaire des systèmes de coordonnées

2.1 – Coordonnées cartésiennes

$$\overrightarrow{grad}\,f = \frac{\partial f}{\partial x}.\overrightarrow{u_x} + \frac{\partial f}{\partial y}.\overrightarrow{u_y} + \frac{\partial f}{\partial z}.\overrightarrow{u_z}$$

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{rot}.\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right).\overrightarrow{u_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right).\overrightarrow{u_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).\overrightarrow{u_z}$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\vec{\Delta}\vec{A} = \Delta A_x.\overrightarrow{e_x} + \Delta A_y.\overrightarrow{e_y} + \Delta A_z.\overrightarrow{e_z}$$

$$\left(\vec{A}.\overrightarrow{\mathrm{grad}}\right)\vec{A} = \left|\begin{array}{l} A_x.\dfrac{\partial A_x}{\partial x} + A_y.\dfrac{\partial A_x}{\partial y} + A_z.\dfrac{\partial A_x}{\partial z} \\ A_x.\dfrac{\partial A_y}{\partial x} + A_y.\dfrac{\partial A_y}{\partial y} + A_z.\dfrac{\partial A_y}{\partial z} \\ A_x.\dfrac{\partial A_z}{\partial x} + A_y.\dfrac{\partial A_z}{\partial y} + A_z.\dfrac{\partial A_z}{\partial z} \end{array}\right.$$

2.2 – Coordonnées cylindriques

$$\overrightarrow{grad}\,f = \frac{\partial f}{\partial r}.\overrightarrow{u_r} + \frac{1}{r}.\frac{\partial f}{\partial \theta}.\overrightarrow{u_\theta} + \frac{\partial f}{\partial z}.\overrightarrow{u_z}$$

$$div\vec{A} = \frac{1}{r}.\frac{\partial (r.A_r)}{\partial r} + \frac{1}{r}.\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{rot}.\vec{A} = \left(\frac{1}{r}.\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right).\overrightarrow{u_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right).\overrightarrow{u_\theta} + \left(\frac{1}{r}.\frac{\partial r.A_\theta}{\partial r} - \frac{1}{r}.\frac{\partial A_r}{\partial \theta}\right).\overrightarrow{u_z}$$

$$\Delta f = \frac{1}{r}.\frac{\partial}{\partial r}\left(r.\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}.\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\vec{\Delta}\vec{A} = \left|\begin{array}{l} \Delta A_r - \dfrac{A_r}{r^2} - \dfrac{2}{r^2}.\dfrac{\partial A_\theta}{\partial \theta} \\ \Delta A_\theta - \dfrac{A_\theta}{r^2} + \dfrac{2}{r^2}.\dfrac{\partial A_r}{\partial \theta} \\ \Delta A_z \end{array}\right.$$

$$\left(\vec{A}.\overrightarrow{\mathrm{grad}}\right)\vec{A} = \left|\begin{array}{l} A_r.\dfrac{\partial A_r}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_r}{\partial \theta} + A_z.\dfrac{\partial A_r}{\partial z} - \dfrac{A_\theta^2}{r} \\ A_r.\dfrac{\partial A_\theta}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_\theta}{\partial \theta} + A_z.\dfrac{\partial A_\theta}{\partial z} + \dfrac{A_r.A_\theta}{r} \\ A_r.\dfrac{\partial A_z}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_z}{\partial \theta} + A_z.\dfrac{\partial A_z}{\partial z} \end{array}\right.$$

2.3 – Coordonnées sphériques

$$\overrightarrow{grad}\,f = \frac{\partial f}{\partial r}.\overrightarrow{u_r} + \frac{1}{r}.\frac{\partial f}{\partial \theta}.\overrightarrow{u_\theta} + \frac{1}{r.\sin\theta}.\frac{\partial f}{\partial \varphi}.\overrightarrow{u_\varphi}$$

$$div\vec{A} = \frac{1}{r^2}.\frac{\partial (r^2.A_r)}{\partial r} + \frac{1}{r.\sin\theta}.\frac{\partial (\sin\theta.A_\theta)}{\partial \theta} + \frac{1}{r.\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\overrightarrow{rot}.\vec{A} = \frac{1}{r.\sin\theta}.\left(\frac{\partial (\sin\theta.A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right).\overrightarrow{u_r} + \frac{1}{r}.\left(\frac{1}{\sin\theta}.\frac{\partial A_r}{\partial \varphi} - \frac{\partial r.A_\varphi}{\partial r}\right).\overrightarrow{u_\theta} + \frac{1}{r}.\left(\frac{\partial r.A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right).\overrightarrow{u_\varphi}$$

$$\Delta f = \frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2.\frac{\partial f}{\partial r}\right) + \frac{1}{r^2.\sin\theta}.\frac{\partial}{\partial \theta}\left(\sin\theta.\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2.\sin^2\theta}.\frac{\partial^2 f}{\partial \varphi^2} \quad \text{ou} \quad \Delta f = \frac{1}{r}.\frac{\partial^2}{\partial r^2}(r.f) + \ldots$$

$$\vec{\Delta}\vec{A} = \left|\begin{array}{l} \Delta A_r - \dfrac{2}{r^2}.\left(A_r + \dfrac{1}{\sin\theta}.\dfrac{\partial A_\theta.\sin\theta}{\partial \theta} + \dfrac{1}{\sin\theta}.\dfrac{\partial A_\varphi}{\partial \varphi}\right) \\ \Delta A_\theta - \dfrac{2}{r^2}.\left(\dfrac{\partial A_r}{\partial \theta} - \dfrac{A_\theta}{2.\sin^2\theta} - \dfrac{\cos\theta}{\sin^2\theta}.\dfrac{\partial A_\varphi}{\partial \varphi}\right) \\ \Delta A_\varphi - \dfrac{2}{r^2.\sin\theta}.\left(\dfrac{\partial A_r}{\partial \varphi} + \dfrac{1}{\tan\theta}.\dfrac{\partial A_\theta}{\partial \varphi} - \dfrac{A_\varphi}{2.\sin\theta}\right) \end{array}\right.$$

$$\left(\vec{A}.\overrightarrow{\mathrm{grad}}\right)\vec{A} = \left|\begin{array}{l} A_r.\dfrac{\partial A_r}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_r}{\partial \theta} + \dfrac{A_\varphi}{r.\sin\theta}.\dfrac{\partial A_r}{\partial \varphi} - \dfrac{A_\varphi^2 + A_\theta^2}{r} \\ A_r.\dfrac{\partial A_\theta}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_\theta}{\partial \theta} + \dfrac{A_\varphi}{r.\sin\theta}\dfrac{\partial A_\theta}{\partial \varphi} + \dfrac{A_r.A_\theta}{r} - \dfrac{A_\varphi^2.\mathrm{cotan}\theta}{r} \\ A_r.\dfrac{\partial A_\varphi}{\partial r} + \dfrac{A_\theta}{r}.\dfrac{\partial A_\varphi}{\partial \theta} + \dfrac{A_\varphi}{r.\sin\theta}\dfrac{\partial A_\varphi}{\partial \varphi} + \dfrac{A_r.A_\varphi}{r} - \dfrac{A_\theta A_\varphi.\mathrm{cotan}\theta}{r} \end{array}\right.$$

3 – Formules intrinsèques

$$\overrightarrow{grad}(f.g) = f.\overrightarrow{grad}.g + g.\overrightarrow{grad}.f \qquad \vec{\nabla}(f.g) = f.\vec{\nabla}g + g.\vec{\nabla}f$$

$$div(f.\vec{A}) = f.div\vec{A} + \vec{A}.\overrightarrow{grad}.f \qquad \vec{\nabla}(f.\vec{A}) = f.\vec{\nabla}\vec{A} + \vec{A}.\vec{\nabla}f$$

$$div(\vec{A}\wedge\vec{B}) = \vec{B}.\overrightarrow{rot}.\vec{A} - \vec{A}.\overrightarrow{rot}.\vec{B} \qquad \vec{\nabla}(\vec{A}\wedge\vec{B}) = \vec{B}.\vec{\nabla}\wedge\vec{A} - \vec{A}.\vec{\nabla}\wedge B$$

$$\overrightarrow{rot}(f.\vec{A}) = f.\overrightarrow{rot}.\vec{A} - \vec{A}\wedge\overrightarrow{grad}f \qquad \vec{\nabla}(f.\vec{A}) = f.(\vec{\nabla}\wedge\vec{A}) - \vec{A}.\wedge\vec{\nabla}f$$

$$\overrightarrow{grad}(\vec{A}.\vec{B}) = \vec{A}\wedge\overrightarrow{rot}.\vec{B} + (\vec{A}.\overrightarrow{grad})\vec{B} + \vec{B}\wedge\overrightarrow{rot}.\vec{A} + (\vec{B}.\overrightarrow{grad})\vec{A}$$

$$\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A}\wedge(\vec{\nabla}\wedge\vec{B}) + (\vec{A}.\vec{\nabla})\vec{B} + \vec{B}\wedge(\vec{\nabla}\wedge\vec{A}) + (\vec{B}.\vec{\nabla})\vec{A}$$

$$\overrightarrow{rot}(\vec{A}\wedge\vec{B}) = (div\vec{B})\vec{A} - (\vec{A}.\overrightarrow{grad})\vec{B} - (div\vec{A})\vec{B} + (\vec{B}.\overrightarrow{grad})\vec{A}$$

$$\vec{\nabla}\wedge(\vec{A}\wedge\vec{B}) = (\vec{\nabla}.\vec{B})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} - (\vec{\nabla}.\vec{A})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A}$$

$$div(\overrightarrow{grad}f) = \Delta f \qquad \vec{\nabla}(\vec{\nabla}f) = \vec{\nabla}^2 f = \Delta f$$

$$\overrightarrow{rot}.\overrightarrow{grad}f = 0 \qquad \vec{\nabla}\wedge\vec{\nabla}f = 0$$

$$div(\overrightarrow{rot}.\vec{A}) = 0 \qquad \vec{\nabla}.(\vec{\nabla}\wedge\vec{A}) = 0$$

$$\boxed{\overrightarrow{rot}(\overrightarrow{rot}\vec{A}) = \overrightarrow{grad}.div\vec{A} - \vec{\Delta}\vec{A}} \qquad \vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

Propriété du produit mixte : $(\vec{A}\wedge\vec{B})\vec{C} = (\vec{C}\wedge\vec{A})\vec{B} = (\vec{B}\wedge\vec{C})\vec{A}$

Double produit vectoriel : $\boxed{\vec{A}\wedge(\vec{B}\wedge\vec{C}) = \vec{B}.(\vec{A}.\vec{C}) - \vec{C}.(\vec{A}.\vec{B})}$

$$\vec{A}\wedge(\vec{B}\wedge\vec{C}) + \vec{C}\wedge(\vec{A}\wedge\vec{B}) + \vec{B}\wedge(\vec{C}\wedge\vec{A}) = 0$$

4 – Relations intégrales

Théorème de Green-Ostrogradski : $\boxed{\oiint_S \vec{A}.\overrightarrow{dS} = \iiint_V div\vec{A}.dV}$

où S est la surface fermée limitant V

Théorème de Stokes-Ampère : $\boxed{\oint_C \vec{A}.\overrightarrow{dM} = \iint_S \overrightarrow{rot}.\vec{A}.\overrightarrow{dS}}$

où S est la surface limitée par la courbe fermée C

Formule du gradient : $\oiint_S f.\overrightarrow{dS} = \iiint_V \overrightarrow{grad}f.dV$

Formule du rotationnel : $\oiint_S \vec{n}\wedge\vec{A}.dS = \iiint_V \overrightarrow{rot}.\vec{A}.dV$